Semi-classical based image reconstruction

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ABSTRACT. Recently, a new signal analysis method based on a semi-classical approach has been proposed [2], [4]. The main idea in this method is to interpret a signal as a potential of a Schrödinger operator and then to use the discrete spectrum of this operator to analyze the signal. In this paper, we are interested in extending this method from one dimension to two dimensions for image processing applications. The efficiency of this approach is shown through some numerical examples. Moreover, the influence of the design parameters in this method is numerically studied.

RÉSUMÉ. Dans cet article, nous nous intéressons à une méthode d'analyse de signaux basée sur une approche semi-classique, développée récemment [2], [4]. L'idée principale de cette méthode consiste à interpréter le signal comme un potentiel de l'opérateur de Schrödinger et à l'analyser en utilisant le spectre discret de cet opérateur. Notre intérêt porte particulièrement sur l'extention de cette méthode au cas bidimensionnel pour des applications en traitement d'images. Nous montrons l'efficacité de la méthode proposée à travers quelques exemples numériques . Par ailleurs, l'influence de certains paramètres, dont dépend cette méthode, est numériquement étudiée.

KEYWORDS : Signal analysis, Image processing, Semi-classical analysis, Schrödinger Operator.

MOTS-CLÉS : Analyse des signaux, Traitement d'images, Analyse semi-classique, Opérateur de Schrödinger.

1. **Acknowledgment** This work was conducted when the first author was visiting the Estimation, Modeling and ANalysis Group at the Computer, Electrical and Mathematical Sciences and Engineering (CEMSE) devision at King Abdullah University of Science and Technology (KAUST). She would like to thank KAUST for its support and generous hospitality.

1. Introduction

Recently, a new signal analysis method based on a semi-classical approach has been proposed (see [2], [4]). We refer to this method as SCSA for **S**emi-**C**lassical **S**ignal **A**nalysis. The main idea of this method is to interpret a signal as a potential of a Schrödinger operator depending on a semi-classical parameter [1]. Then, the signal can be characterized using the discrete spectrum of the Schödinger operator. Promising results have been obtained when applying the SCSA method to the analysis of arterial blood pressure signals [2], [5] and to the analysis of the proformance of turbomachinery. Moreover, it has been shown that the SCSA method can cope with noisy signals, making this method a potential tool for denoising. The aim of this paper is to extend the SCSA method from one dimensional case to two dimensional case.

2. Preliminary

In this section, we propose to recall the idea behind of the SCSA method [2], [4]. Let us consider the following Schrödinger operator:

$$H_{1,h}(V_1) := -h^2 \frac{d^2}{dx^2} - V_1, \tag{1}$$

where $h \in \mathbb{R}_+^*$ is the semi-classical parameter [1], and V_1 is a positive real valued function belonging to $\mathcal{C}^\infty(\Omega_1)$ where $\Omega_1 \subset \mathbb{R}$ is a compact. Then, the potential V_1 can be estimated using the following proposition.

Proposition 2.1 *[2] Let $V_1 \in \mathcal{C}^\infty(\Omega_1)$ be positive real valued function, where $\Omega_1 \subset \mathbb{R}$ is a compact. Then, V_1 can be estimated using the following formula: $\forall\, x \in \Omega_1$,*

$$V_{1,h,\gamma,\lambda}(x) := -\lambda + \left(\frac{h}{L_{1,\gamma}^{cl}} \sum_{n=1}^{N_h^\lambda} (\lambda - \lambda_{n,h})^\gamma \psi_{n,h}^2(x) \right)^{\frac{2}{2\gamma+1}}, \tag{2}$$

where $h \in \mathbb{R}_+^*$, $\gamma \in \mathbb{R}_+$, $\lambda \in \mathbb{R}_-^*$, and $L_{1,\gamma}^{cl}$ is the universal semi-classical constant given by: $L_{1,\gamma}^{cl} = \frac{1}{2\sqrt{\pi}} \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\frac{3}{2})}$, where Γ is the gamma function. Moreover, $\lambda_{n,h}$ are the negative eigenvalues of $H_{1,h}(V_1)$ with $\lambda_{1,h} < \cdots < \lambda_{N_h^\lambda,h} < \lambda$, N_h^λ is the number of negative eigenvalues smaller than λ, and $\psi_{n,h}$ are the associated L^2-normalized eigenfunctions such that: $H_{1,h}(V_1)\,\psi_{n,h} = \lambda_{n,h}\psi_{n,h}$.

The efficiency of the proposed signal estimation method and the influence of the design parameters λ, γ and h have been studied in [2]. In particular, as it is described in [2] and [4], the semi-classical parameter h plays a key role in this approach. In fact, when h decreases, the estimation $V_{h,\gamma,\lambda}$ improves. Since the study of the Schrödinger operator in the case where h tends to 0 is referred to the semi-classical analysis [1], this justifies the name **S**emi-**C**lassical **S**ignal **A**nalysis that we give to this approach [2], [4].

Let us point out that the formula given in (2) is still applicable in the case where $\lambda = 0$, and it gives good results even if λ is out of the range of definition. It has been used for example in the analysis of the arterial blood pressure signal in [5].

3. Generalization of the SCSA method to two dimensions

In this section, we propose to generalize formula (2) from one dimension to two dimensions. From now on, we consider the following $2D-$Schrödinger operator associated to a potential V_2:

$$H_{2,h}(V_2) := -h^2\Delta - V_2, \tag{3}$$

where $\Delta := \frac{d^2}{dx^2} + \frac{d^2}{dy^2}$ is the $2D$-Laplacien operator, $h \in \mathbb{R}_+^*$ is the semi-classical parameter [1], and V_2 is a positive real valued function belonging to $\mathcal{C}^\infty(\Omega_2)$ where $\Omega_2 \subset \mathbb{R}^2$ is a compact.

Let us recall that, as described in the previous section, the main idea behind the SCSA method in one dimension is to use the negative eigenvalues and the associated eigenfunctions of the operator $H_{1,h}(V_1)$ (equation (1)) to estimate the potential V_1. In order to estimate the potential V_2 by using a similar approach, we focus on the following spectral problem:

$$H_{2,h}(V_2)\psi_{k,h} = \lambda_{k,h}^{x,y}\psi_{k,h}, \tag{4}$$

where $\lambda_{k,h}^{x,y}$ and $\psi_{k,h}$, for $k = 1, \cdots, K_h^\lambda$, refer to the negative eigenvalues and the associated L^2-normalized eigenfunctions respectively.

We propose to use the separation of variables to calculate the eigenfunctions $\psi_{k,h}$. Hence, by taking $\psi_{k,h}(x,y) = \varphi_{n,h}(x)\phi_{m,h}(y)$ in (4), for $n = 1, \cdots, N_h^\lambda$ and $m = 1, \cdots, M_h^\lambda$ with $K_h^\lambda = N_h^\lambda M_h^\lambda$, we obtain: $\forall\, (x,y) \in \Omega_2$,

$$-h^2\Delta\left\{\varphi_{n,h}(x)\phi_{m,h}(y)\right\} - V_2(x,y)\varphi_{n,h}(x)\phi_{m,h}(y) = \lambda_{k,h}^{x,y}\,\varphi_{n,h}(x)\phi_{m,h}(y). \tag{5}$$

Then, it implies that: $\forall\, (x,y) \in \Omega_2$,

$$\begin{aligned}\lambda_{k,h}^{x,y}\varphi_{n,h}(x)\phi_{m,h}(y) = &\left(-h^2\frac{d^2}{dx^2}\varphi_{n,h}(x) - \frac{1}{2}V_2(x,y)\varphi_{n,h}(x)\right)\phi_{m,h}(y)\\ &+\left(-h^2\frac{d^2}{dy^2}\phi_{m,h}(y) - \frac{1}{2}V_2(x,y)\phi_{m,h}(y)\right)\varphi_{n,h}(x).\end{aligned} \tag{6}$$

Let us consider the following eigenvalues problem: $\forall\,(x,y)\in\Omega_2$,

$$H^x_{1,h}\left(\frac{1}{2}V_2(\cdot,y)\right)\varphi_{n,h}(x)=\lambda^x_{n,h}\varphi_{n,h}(x), \tag{7}$$

$$H^y_{1,h}\left(\frac{1}{2}V_2(x,\cdot)\right)\phi_{m,h}(y)=\lambda^y_{m,h}\phi_{m,h}(y), \tag{8}$$

where $\lambda^x_{n,h}$ (resp. $\lambda^y_{m,h}$) are the negative eigenvalues of the operator $H^x_{1,h}(\frac{1}{2}V_2(\cdot,y))$ (resp. $H^y_{1,h}(\frac{1}{2}V_2(x,\cdot))$), $n=1,\cdots,N^\lambda_h$, (resp. $m=1,\cdots,M^\lambda_h$), and $\varphi_{n,h}$ (resp. $\phi_{m,h}$) are the associated L^2-normalized eigenfunctions. Then, by using (7) and (8) in (6), we get: $(\lambda^x_{n,h}+\lambda^y_{m,h})\,\varphi_{n,h}(x)\phi_{m,h}(y)=\lambda^{x,y}_{k,h}\varphi_{n,h}(x)\phi_{m,h}(y)$, $\forall\,(x,y)\in\Omega_2$.

Consequently, we obtain the following relation

$$\lambda^{x,y}_{k,h}=\lambda^x_{n,h}+\lambda^y_{m,h}, \tag{9}$$

for $n=1,\cdots,N^\lambda_h$, $m=1,\cdots,M^\lambda_h$ and $k=1,\cdots,N^\lambda_h M^\lambda_h$.

Finally, let us recall that the coefficient $\frac{h}{L^{cl}_{1,\gamma}}$ given in Proposition 2.1 is related to some Riesz means connected to a Lieb-Thirring's conjecture (see [3]). Then we refer to the n dimensional formulation of the Riesz means to find the coefficient of the sum in two dimensions. In particular, we show that this coefficient is given by $\frac{h^2}{L^{cl}_{2,\gamma}}$, where the suitable universal semi-classical constant $L^{cl}_{2,\gamma}$ is given by $L^{cl}_{2,\gamma}=\frac{1}{2^2\pi}\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+2)}$.

Consequently, inspired from the idea of tensor product and by using a similar way for obtaining Proposition 2.1, we can obtain the following proposition.

Proposition 3.1 *Let V_2 be a positive real valued function belonging to $\mathcal{C}^\infty(\Omega_2)$ where $\Omega_2\subset\mathbb{R}^2$ is a compact. Then, V_2 can be estimated by the following formula: $\forall\,(x,y)\in\Omega_2$,*

$$V_{2,h,\gamma,\lambda}(x,y):=-\lambda+\left(\frac{h^2}{L^{cl}_{2,\gamma}}\sum_{m=1}^{M^\lambda_h}\sum_{n=1}^{N^\lambda_h}\Big(\lambda-(\lambda^x_{n,h}+\lambda^y_{m,h})\Big)^\gamma\varphi^2_{n,h}(x)\phi^2_{m,h}(y)\right)^{\frac{2}{2\gamma+1}}, \tag{10}$$

where $h\in\mathbb{R}^*_+$, $\gamma\in\mathbb{R}_+$, $\lambda\in\mathbb{R}_-$, and $L^{cl}_{2,\gamma}$ is the suitable universal semi-classical constant. Moreover, $\lambda^x_{n,h}$ (resp. $\lambda^y_{m,h}$) are the negative eigenvalues of the operator $H^x_{1,h}(\frac{1}{2}V_2(\cdot,y))$ (resp. $H^y_{1,h}(\frac{1}{2}V_2(x,\cdot))$ with $\lambda^x_{1,h}<\cdots<\lambda^x_{N^\lambda_h,h}<\lambda$ (resp. $\lambda^y_{1,h}<\cdots<\lambda^y_{M^\lambda_h,1}<\lambda$), N^λ_h (resp. M^λ_h) is the number of the negative eigenvalues smaller than λ, and $\psi_{n,h}$ (resp. $\phi_{m,h}$) are the associated L^2-normalized eigenfunctions.

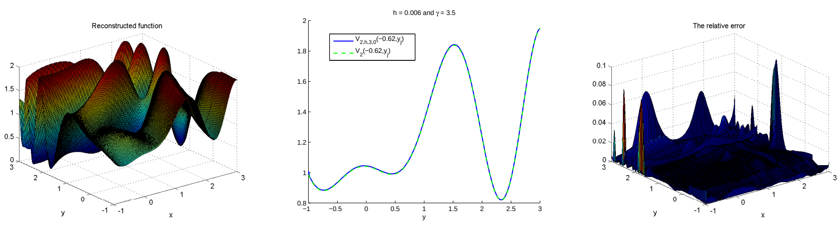
4. Some numerical examples

In order to validate formula (10), we have performed numerical tests on two examples. Note that, in this section, we will take $\lambda = 0$ in formula (10).

Example 1. In this example, we consider the following function:

$$V_2(x,y) = \sin(\frac{1}{2}x^2 + \frac{1}{4}y^2 + 3)\, \cos(2x + 1 - e^y) + 1. \tag{11}$$

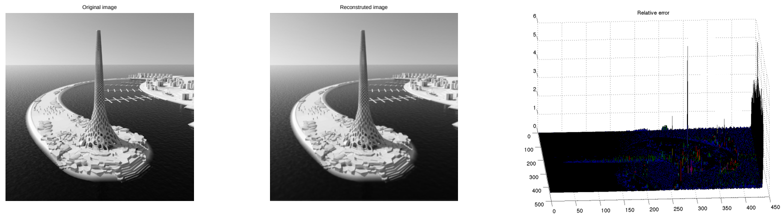
Then, we assume that V_2 is given in a discrete case where $x_i = iT_s$ and $y_j = jT_s$ for $i, j = -50, \cdots, 150$ with $T_s = 0.02$. We can remark that since the number of negative eigenvalues increases while h decreases, in practice we can not take h very small as described in [2] and [4]. So, before estimating V_2, we study the influence of the design parameters h and γ. By taking different values of h and γ, and by estimating the variation of the mean square errors between V_2 and the estimation $V_{2,h,\gamma,0}$, we found that there is a minimum at $h = \frac{6}{10^3}$, and $\gamma = 3.5$. Then, we can estimate V_2 by using $V_{2,h,\gamma,0}$ with these optimal parameter values (see Figure 1(a)). In particular, we show in Figure 1(b) the original signal $V_2(-0.62, y_j)$ and the estimation $V_{2,h,\gamma,0}(-0.62, y_j)$ with $y_j = -1, -0.98, \cdots, 3$ and in Figure1(c) we show the relative error between the function and its estimation.



(a) (b) (c)

Figure 1. *Example 1:* $V_2(x,y) = \sin(\frac{1}{2}x^2 + \frac{1}{4}y^2 + 3)\, \cos(2x+1-e^y)+1$. (a) $V_{2,h,\gamma,0}(x_i, y_j)$. (b) $V_2(-0.62, y_j)$, $V_{2,h,\gamma,0}(-0.62, y_j)$ *with* $y_j = -1, -0.98, \cdots, 3$. (c) *The relative error between the real function and its estimation.*

Example 2. In this example, we consider a 440×440 pixels image, see Figure 2(a). One can note the good reconstruction of this image in Figure 2(b), for $h = 0.3$ and $\gamma = 3.5$, and the associated relative error in Figure 2(c).



(a) (b) (c)

Figure 2. *Example 2: The beacon in KAUST. (a) Original image. (b) Reconstructed image. (c) The relative error between the original and reconstructed images.*

5. Conclusion

In this paper, we have generalized a recent semi-classical signal analysis method from one dimensional case to two dimensional case for image analysis. The efficiency of the proposed method has been shown through some numerical examples. Moreover, the influence of design parameters in the method has been numerically studied. Accurate error analysis will be given in a future work.

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